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THE STRUCTURE OF GENERALIZED MORSE MINIMAL SETS ON n SYMBOLS

BY

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ABSTRACT. A class of bisequences on n symbols is constructed which includes the generalized Morse sequences introduced by Keane. Those which give rise to strictly ergodic sets are characterized, and the spectrum of the shift operator on these systems is investigated. It is shown that in certain cases the shift operator has partly discrete and partly continuous spectrum. The theorems generalize results of Keane on generalized Morse sequences and a theorem of Kakutani regarding a particular strictly transitive sequence on four symbols. Another special case yields information on the spectrum of certain substitution minimal sets.

Introduction. One of the most important sources of examples of interesting minimal sets has been symbolic dynamics; often, by using a finite number of symbols to construct infinite sequences with certain combinatorial properties, one can generate symbolic minimal sets which have desired dynamical and ergodic properties and which shed light on more general dynamical systems. A particularly fruitful example is the Morse sequence

$$x = 0110100110010110\ldots.$$

This was introduced by Morse and independently by Thue, and its properties are fairly well known [3], [4], [8], [9]. It has served as a prototype for a number of other constructions, including substitution sequences [1], [2], [7], [10] and a class of sequences on two symbols, introduced by Keane [5], called generalized Morse sequences.

In [5], Keane suggests extending his construction to more than two symbols and poses the problem of carrying out an analysis of the more general systems with regard to strict ergodicity and the spectrum of the shift operator. In [11], one such construction is described, and some of the topological properties of the resulting minimal sets are discussed. In this paper we consider a slightly less general construction than that in [11], one which yields systems which are more amenable to measure-theoretic analysis.

The first main result (Theorem 4) is a characterization of those generalized

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Morse sequences ω on n symbols which yield strictly ergodic sets. Secondly, if (Ω, T) is the minimal set arising from such a sequence ω, it will be shown (Theorem 14) that in certain cases, the space of square-integrable functions on Ω may be decomposed into the direct sum of two invariant subspaces Γ_0 and Γ, on which the shift operator has discrete and continuous spectrum, respectively. Some of the supporting lemmas carry over from [5] to the more general case with only minor changes, while others require more detailed investigation. The principal ingredient in the proofs of these theorems is a lemma (Lemma 2) concerning convergence of certain sequences of successive convolutions on a finite Abelian group. Using this result, we are able to give a somewhat more unified answer to these two questions. Thus the paper generalizes, and in some respects clarifies, the results of [5].

In several other special cases the construction leads to sequences which have been studied previously. One case allows us to derive a result of Kakutani [4] concerning a sequence of four symbols which can be generated by considering, for each k, the last nonzero digit in the decimal expansion of $k!$. Another special case yields a class of substitution minimal sets; our results enable us to describe the spectrum of certain of these sets, and thus to generalize a portion of [1].

The first section of the paper describes the construction and gives some of the necessary preliminaries from symbolic dynamics. For a more complete reference, the reader is referred to [3] or [5].

1. Construction and preliminaries. Let n be an integer greater than 1, and let G be an (additive) Abelian group of order n. We shall consider sequences on n symbols, where for convenience the symbol set is taken to be G. For $k > 1$, B_k will denote the set of k-blocks over G–that is, blocks of the form $A = A(0)A(1) \cdots A(k-1)$, where each $A(i)$ is an element of G. X will denote the set of sequences over G (functions from the nonnegative integers to G) and Ω the set of bisequences over G. If $A \in B_k$, $C \in B_m$, then $AC \in B_{k+m}$ is defined by $AC = A(0) \cdots A(k-1)\, C(0) \cdots C(m-1)$. For x an element of B_k, X, or Ω, we will denote by $x(j,m)$ the m-block $x(j)x(j+1) \cdots x(j+m-1)$. If $A \in B_m$ and x is an element of B_k, X, or Ω, we say A appears in x if for some j, $x(j,m) = A$. If $A \in B_k$, let $L(A) = k$. Finally, if $g \in G$, we shall use σ_g to denote the function from any of the sets B_k, X, or Ω to itself defined by

$$\sigma_g(A)(j) = A(j) + g.$$

Thus $\{\sigma_g : g \in G\}$ is a subgroup of order n of the group of permutations of G.

Now if $A \in B_j$, $C \in B_k$, we define

$$A \times C = \sigma_{C(0)}A\sigma_{C(1)}A \cdots \sigma_{C(k-1)}A \in B_{jk}.$$

We observe that the operation $\times$ is associative. For each integer $t \geq 0$, let λ_t be an integer greater than 1 and let b_t be an element of B_{λ_t} such that if $t \geq 1$, $b_t(0) = 0$. Then we define an element x of X as follows:

$$x = b_0 \times b_1 \times \cdots .$$

For each $t \geq 0$, let $n_t = \lambda_0\lambda_1 \cdots \lambda_t$, and let $c_t = b_0 \times b_1 \times \cdots \times b_t$, the initial n_t-block of x. We assume at this point that each of the sequences $b_t \times b_{t+1} \times \cdots$ contains every symbol in G; we may therefore assume, by grouping some of the b_i's if necessary, that each block b_t contains every symbol in G.

We let T: $\Omega \to \Omega$ denote the shift transformation: $(Ty)(k) = y(k+1)$ $(y \in \Omega, k \in Z)$. At this point, using Lemma 1 and Proposition 2 of [11], we assume x is nonperiodic, and we choose a fixed almost periodic point $\omega \in \Omega$ with $\omega(k) = x(k)$ $(k \geq 0)$. We denote by Ω_x the orbit-closure of ω under T; then (Ω_x, T) is a symbolic minimal set. The sequence ω is called a *generalized Morse sequence*, and (Ω_x, T) is called a generalized Morse minimal set.

In [11] the construction is somewhat more general in that the functions σ_g are elements of any group G of order n of permutations of the elements of G.

We shall need a combinatorial lemma which is proved in [11].

LEMMA 1. *For any $t \geq 0$, there exists k so that any k-block A of x has the property that whenever $A = x(j,k) = x(m,k)$, then $j = m$* (mod n_t).

An immediate consequence of Lemma 1 is the fact that for each t, every point $y \in \Omega_x$ has a unique representation in the form $y = \cdots \sigma_{g_{-1}}(c_t)\sigma_{g_0}(c_t)\sigma_{g_1}(c_t) \cdots$; or equivalently, for each t and each $y \in \Omega_x$, there is a unique k $(0 \leq k < n_t)$ such that $y(k + jn_t, n_t) = \sigma_{g_j}(c_t)$ for every $j \in Z$.

We may therefore define, for each $t \geq 0$ and each k satisfying $0 \leq k < n_t$,

$$D_k^t = \{ y \in \Omega_x : y(-k + jn_t, n_t) = \sigma_{g_j}(c_t) \text{ for each } j \in Z \}.$$

It may be seen that for each t, $\{D_0^t, D_1^t, \ldots, D_{n_t-1}^t\}$ is a partition of Ω_x into open-and-closed subsets, and

$$T(D_i^t) = D_{i+1}^t \quad (i < n_t - 1); \qquad T(D_{n_t-1}^t) = D_0^t.$$

Moreover, it is clear that each σ_g is a homeomorphism from Ω_x onto itself which commutes with T, and

$$\sigma_g(D_k^t) = D_k^t \qquad (t \geq 0, 0 \leq k < n_t).$$

If A is any subset of Ω_x, we will let 1_A denote the characteristic function of A.

2. Strict ergodicity of generalized Morse minimal sets. In this section we characterize those generalized Morse minimal sets which are strictly ergodic—or equivalently, those generalized Morse sequences which are strictly transitive. The analysis is based primarily upon a lemma concerning convolutions on finite Abelian groups, the statement of which requires some notation.

Let f be a nonnegative function on G. We shall denote by $\int_G f$ the number $|G|^{-1}\Sigma_{x\in G}f(x)$. (Here $|G|$ represents the cardinality of G.) For any subgroup H of G of index $(G:H)$, f induces a function f^H on the group G/H as follows:

$$f^H([x]) = |H|^{-1} \sum_{h\in H} f(x+h).$$

We let $m(f,H)$ denote the maximum value of f^H, and we note that

$$(G:H)^{-1}m(f,H) \leqslant \int_G f.$$

Now if f and g are functions on G, the convolution $f * g$ is defined by

$$f * g(x) = |G|^{-1} \sum_{y\in G} f(y)g(x-y).$$

The operation of convolution is commutative and associative, $\int_G f * g = (\int_G f)(\int_G g)$, and we have, for any subgroup H of G,

$$(f * g)^H = f^H * g^H.$$

LEMMA 2 [12]. *Let $f_0, f_1, \ldots$ be a sequence of functions on G satisfying $f_i \geqslant 0$ and $\int_G f_i = 1$ $(i \geqslant 0)$. Then the sequence $f_t, f_t * f_{t+1}, f_t * f_{t+1} * f_{t+2}, \ldots$ converges to* 1 *for every t if and only if, for every proper subgroup H of G,*

$$\prod_{t=0}^{\infty} \left((G:H)^{-1}m(f_t,H)\right) = 0.$$

REMARK. This is a special case of a more general result concerning successive convolutions of probability measures on an arbitrary zero-dimensional, compact, Abelian group. The statement in the general case must be modified slightly, and details are given in [12].

We now apply Lemma 2 to the problem of strict ergodicity of the minimal set (Ω_x,T). For $A \in B_k$, $C \in B_j$, let $r_C(A) = m/k$, where m is the number of appearances of C in A. In particular, for $g \in G$, $r_g(A)$ is the frequency with which the symbol g appears in A. If $y \in X$ and $C \in B_j$, we let $r_C(y) = \lim_{k\to\infty} r_C(y(0,k))$, whenever this limit exists. Clearly, the limit exists if and only if all the limits $\lim_{k\to\infty} r_C(y(t,k))$ exist and are equal; we say $r_C(y)$ exists uniformly if these limits exist uniformly in t. The sequence y is *strictly transitive* if and only if, for every block A appearing in y, $r_A(y)$ exists uniformly.

LEMMA 3 [13]. *(Ω_x,T) is strictly ergodic if and only if $x = b_0 \times b_1 \times \cdots$ is strictly transitive.*

Now for each $t \geqslant 0$, define f_t on G by

$$f_t(g) = nr_g(b_t).$$

Then $f_t \geq 0$ and $\int_G f_t = 1$, and it may be verified that

(1) $$f_t * f_{t+1} * \cdots * f_{t+k}(g) = nr_g(b_t \times b_{t+1} \times \cdots \times b_{t+k}).$$

With notation as above, we have the following:

THEOREM 4. *(Ω_x, T) is strictly ergodic if and only if, for every proper subgroup H of G, $\prod_{t=0}^{\infty}((G:H)^{-1}m(f_t,H)) = 0$.*

REMARK. If $G = Z_2$, then $H = \{0\}$ is the only proper subgroup, and the statement reduces to Lemma 3 of [5].

The theorem follows from Lemmas 2 and 3 and the following.

LEMMA 4.1. *x is strictly transitive if and only if $r_g(b_{t+1} \times b_{t+2} \times \cdots) = 1/n$ for every $g \in G$ and $t \geq 0$.*

PROOF. Suppose this condition is satisfied; let A be any m-block appearing in x, and let $\varepsilon > 0$. We wish to show that for sufficiently large p, any two p-blocks E and F of x satisfy $|r_A(E) - r_A(F)| < \varepsilon$. We may choose t and k_0 so large that if C is any block of x of the form

$$C = C_1\sigma_{g_1}(c_t) \cdots \sigma_{g_k}(c_t)C_2 \qquad (0 \leq L(C_1), L(C_2) < n_t, k \geq k_0),$$

then

$$\left|r_A(C) - k^{-1}\left(r_A(\sigma_{g_1}(c_t)) + \cdots + r_A(\sigma_{g_k}(c_t))\right)\right| < \varepsilon/4.$$

Now since $r_g(b_{t+1} \times b_{t+2} \times \cdots) = 1/n$ for each g, we may choose $j_0 \geq k_0$ so that if $j \geq j_0$, then for any j-block D of $b_{t+1} \times b_{t+2} \times \cdots, |r_g(D) - 1/n| < \varepsilon/4$ for each g. Let E be any p-block of x, where $p \geq (j_0 + 2)n_t$. Then

$$E = E_1\sigma_{g_1}(c_t) \cdots \sigma_{g_q}(c_t)E_2,$$

where $0 \leq L(E_1), L(E_2) < n_t$ and $q \geq j_0$. On the one hand,

$$\left|r_A(E) - q^{-1}\left(r_A(\sigma_{g_1}(c_t)) + \cdots + r_A(\sigma_{g_q}(c_t))\right)\right| < \varepsilon/4;$$

on the other hand,

$$\left|q^{-1}\left(r_A(\sigma_{g_1}(c_t)) + \cdots + r_A(\sigma_{g_q}(c_t))\right) - n^{-1}\sum_{g\in G} r_A(\sigma_g(c_t))\right|$$
$$< (\varepsilon/4)\sum_{g\in G} r_A(\sigma_g(c_t)) \leq \varepsilon/4.$$

It follows that if E and F are both p-blocks and $p \geq (j_0 + 2)n_t$, then $|r_A(E) - r_A(F)| < \varepsilon$.

Now suppose x is strictly transitive. It is sufficient to show that for $g \in G$, $r_g(b_1 \times b_2 \times \cdots) = 1/n$. If this is not so, then for some $g,h \in G$ and $\varepsilon > 0$, we may assume that for each s,

$$|r_g(b_1 \times b_2 \times \cdots \times b_s) - r_h(b_1 \times \cdots \times b_s)| > \varepsilon,$$

Choose t, using Lemma 1, so that whenever $x(p, n_t) = x(q, n_t)$, then $p \equiv q$ (mod n_0). Let $A = \{C \in B_{n_t}: C(0) = g,\ C = x(kn_0, n_t)$ for some $k\}$. For any m-block D, denote by $r_A(D)$ the sum $\sum_{C \in A} r_C(D)$. Then because of the way we have chosen t, we see that for any $\delta > 0$,

$$|r_A(c_s) - n_0^{-1} r_g(b_1 \times \cdots \times b_s)| < \delta,$$

and therefore

$$|r_A(\sigma_{g-h}(c_s)) - n_0^{-1} r_h(b_1 \times \cdots \times b_s)| < \delta,$$

for sufficiently large s. Now since x is strictly transitive, $|r_A(c_s) - r_A(\sigma_{g-h}(c_s))|$ can be made arbitrarily small by choosing s sufficiently large; but since

$$|r_g(b_1 \times \cdots \times b_s) - r_h(b_1 \times \cdots \times b_s)| > \varepsilon,$$

we obtain a contradiction by choosing δ sufficiently small. (Q.E.D.)

3. **The spectrum of** (Ω_x, T). For the remainder of the paper, we assume that the sequence $x = b_0 \times b_1 \times \cdots$ is strictly transitive and that ω is a fixed extension of x as before. Let m_x be the unique T-invariant Borel probability measure on Ω_x. Then if $C \in B_k$, and $A = \{y \in \Omega_x: y(t,k) = C\}, m_x(A)$ is given by the formula $m_x(A) = r_C(x)$. Since $r_C(x) = r_{\sigma_g(C)}(x)$ for every $g \in G$, each of the maps $\sigma_g: \Omega_x \to \Omega_x$ is measure-preserving.

We now consider the unitary operator (also denoted by T) on $L^2(\Omega_x, m_x)$ induced by T. Suppose that the finite Abelian group G is isomorphic to the direct product of cyclic groups

$$G = Z_{m_1} \times Z_{m_2} \times \cdots \times Z_{m_k},$$

so that $m_1 m_2 \cdots m_k = n$. We denote by g_i the element $(0, \ldots, 0,1,0, \ldots, 0)$ of G, where 1 is in the ith coordinate. In addition, let $\alpha_i = e(1/m_i) = \exp(2\pi i/m_i)$, and for $0 \leqslant j \leqslant m_i - 1$, let

$$\Lambda(i,j) = \{f \in L^2: f \circ \sigma_{g_i} = \alpha_i^j f\}.$$

Now for $g = (j_1, j_2, \ldots, j_k) \in G$, let

$$\Gamma_g = \Lambda(1,j_1) \cap \Lambda(2,j_2) \cap \cdots \cap \Lambda(k,j_k).$$

Since each σ_g commutes with T, each Γ_g is a T-invariant subspace, and it is not difficult to verify that $L^2(\Omega_x, m_x)$ may be decomposed into the direct sum

$$L^2(\Omega_x, m_x) = \bigoplus_{g \in G} \Gamma_g.$$

It is also clear that if $f \in \Gamma_g$, $f' \in \Gamma_h$, and $ff' \in L^2$, then $ff' \in \Gamma_{g+h}$. We denote by Γ the subspace $\Gamma = \bigoplus_{g \neq 0} \Gamma_g$.

THEOREM 5. *For each $t \geq 0$, $e(1/n_t)$ is an eigenvalue of T corresponding to the continuous eigenfunction $f_t = \sum_{j=0}^{n_t-1} e(j/n_t) 1_{D_j t}$ belonging to Γ_0. T has pure point spectrum on Γ_0, and the eigenvalue group of T on Γ_0 is*

$$\Delta = \{ e(j/n_t) : t \geq 0, 0 \leq j < n_t \}.$$

PROOF. In the case $G = Z_2$, this is a combination of Theorems 5 and 6 of [5]; the proofs given there hold in the general case with only minor changes. (Q.E.D.)

Let p be a prime factor of n. We denote by F_p the set

$$F_p = \{ g \in G : p | m \text{ whenever } mg = 0 \}.$$

It is easy to see that $G \setminus \{0\} = \bigcup F_p$, where the union is over all prime factors p of n.

LEMMA 6. *Let p be a prime factor of n, and let $g \in F_p$. If there is an eigenfunction for T in Γ_g, then for some t, $e(1/(pn_t))$ is an eigenvalue of T corresponding to an eigenfunction in Γ_h, for some $h \in F_p$, and $p \nmid \lambda_s$ for $s > t$.*

PROOF. Let $Tf = \theta f$, where $f \in \Gamma_g$ and $g \in F_p$. By taking an appropriate multiple of g, and raising f to that power, we may assume $rg = 0$ if and only if $p | r$. Since $f^p \in \Gamma_0$, we have $\theta^p \in \Delta$ (and $\theta \notin \Delta$), so that $\theta = e(s/(pn_t))$ for some $t \geq 0$, where $p \nmid s$. Choose integers u and v so that $us + vp = 1$. Then $p \nmid u$, and therefore if $h = ug$, then $h \in F_p$. Now $e(vp/(pn_t)) \in \Delta$, and therefore $e(1/(pn_t))$ is an eigenvalue corresponding to $f^u \in \Gamma_h$. If $p | \lambda_s$ for some $s > t$, then by writing $n_s = n_t \lambda_s k$, we see that $e(1/(pn_t)) = e(\lambda_s k/(pn_s)) \in \Delta$; but this is impossible, because the eigenvalue $e(1/(pn_t))$ is simple. (Q.E.D.)

COROLLARY 7. *If $p | \lambda_s$ for infinitely many s, then T has continuous spectrum on Γ_g for every $g \in F_p$. If every prime factor of n divides infinitely many λ_i's, then T has continuous sepctrum on Γ.*

PROOF. The first statement is clear. The second follows from the fact that if $f = \sum_{g \neq 0} f_g$ is an eigenfunction for T and $f_g \in \Gamma_g$, then each nonzero f_g is also an eigenfunction. (Q.E.D.)

LEMMA 8. *For a prime factor p of n, the following statements are equivalent.*

(a) *There is an eigenfunction in Γ_g for some $g \in F_p$.*

(b) *$e(1/(pn_t))$ is an eigenvalue for some t, with $p \nmid \lambda_s$ for $s > t$.*

(c) *$e(1/p)$ is an eigenvalue for T_y on (Ω_y, T_y), where $y = b_{t+1} \times b_{t+2} \times \cdots$ for some t, and $p \nmid \lambda_s$ for $s > t$.*

(d) *There is an eigenfunction for T_y in the subspace Γ_g of $L^2(\Omega_y, m_y)$, for some y as in* (c) *and some $g \in F_p$.*

PROOF. We have already seen that (a) implies (b); for the converse, we note that $e(1/(pn_t)) \not\in \Delta$, since $p \nmid \lambda_s$ for $s > t$. If $e(1/(pn_t))$ corresponds to the eigenfunction f, then we may assume $f \in \Gamma_g$ for some g. It follows that $g \neq 0$, that $f^p \in \Gamma_0$, and therefore that $g \in F_p$. (c) and (d) are equivalent by an almost identical argument. It remains to verify that (b) and (c) are equivalent.

Suppose $e(1/(pn_t))$ is an eigenvalue corresponding to $f \in \Gamma_g$, where $g \in F_p$ and $p \nmid \lambda_s$ for $s > t$. Let $y = b_{t+1} \times b_{t+2} \times \cdots$. Then we define $\psi\colon \Omega_y \to D_0^t$ by

$$\psi(z)(in_t, n_t) = \sigma_{z(i)}(c_t).$$

ψ is an isomorphism from the strictly ergodic set (Ω_y, T_y, m_y) to $(D_0^t, T^{n_t}, n_t m_x)$ (where $n_t m_x$ is the suitably normalized restriction of m_x to D_0^t); and ψ commutes with σ_h for every h. Now if f_1 denotes the restriction of f to D_0^t, then f_1 is an eigenfunction for T^{n_t} corresponding to the eigenvalue $e(1/p)$. Therefore, $f_1 \circ \psi$ is an eigenfunction for T_y corresponding to $e(1/p)$.

Conversely, if f_1 is an eigenfunction for T_y corresponding to $e(1/p)$, and $f_1 \in \Gamma_g$ for some $g \in F_p$, then $f = f_1 \circ \psi^{-1}$, defined on D_0^t, is an eigenfunction for T^{n_t} with eigenvalue $e(1/p)$. If we extend this function inductively to a function f defined on $\Omega_x = D_0^t \cup D_1^t \cup \cdots \cup D_{n_t-1}^t$ by the formula

$$f(Tz) = e(1/(pn_t))f(z),$$

then $f \in \Gamma_g$ and $Tf = e(1/(pn_t))f$. (Q.E.D.)

COROLLARY 9. *Suppose $p \nmid \lambda_s$ for $s > t$. Let $y = b_{t+1} \times b_{t+2} \times \cdots$. Then T has continuous spectrum on Γ_g for every $g \in F_p$ if and only if the same is true for T_y.*

It follows that in analyzing the spectrum of T on Γ_g for $g \in F_p$, we may restrict ourselves to the case in which $p \nmid \lambda_s$ $(s \geq 0)$.

LEMMA 10. *Let p be a prime factor of n, with $p \nmid \lambda_t$ $(t \geq 0)$. Then T has continuous spectrum on Γ_g for every $g \in F_p$ if and only if T^p is ergodic with respect to m_x.*

PROOF. If there is an eigenfunction in Γ_g for some $g \in F_p$, then since $p \nmid \lambda_s$ for any s, there is an eigenfunction f in Γ_h, for some $h \in F_p$, with eigenvalue $e(1/p)$. Thus $T^pf = f$, and therefore T^p cannot be ergodic.

If T^p is not ergodic, then there is a T^p-invariant set $A \subset \Omega_x$ with $0 < m_x(A) < 1$. Let

$$f = 1_A + e(1/p)1_{T(A)} + \cdots + e(1 - 1/p)1_{T^{p-1}(A)}.$$

Then $Tf = e(1/p)f$. If we can show that f is not identically zero, then since $e(1/p) \not\in \Delta$, it will follow that there is an eigenfunction in Γ_g for some $g \in F_p$. Now since A is not T-invariant, $m_x(T(A) \setminus A) > 0$. But from the fact that $e(1/p)$, $e(2/p), \ldots, e(1-1/p)$ are linearly independent over the rationals (which is true since p is prime), we conclude that f is never zero on the set $T(A) \setminus A$. (Q.E.D.)

THEOREM 11. *T has continuous spectrum on Γ if and only if, for every prime factor p of n, either $p|\lambda_s$ for infinitely many s, or, for some t with $p \nmid \lambda_s$ for $s > t$, T_y^p is ergodic, where $y = b_{t+1} \times b_{t+2} \times \cdots$.*

PROOF. This is a result of combining Lemmas 6 through 10.(Q.E.D.)

LEMMA 12. *For any $p \geqslant 2$, T^p is strictly ergodic if and only if T^p is ergodic with respect to the measure m_x.*

PROOF. This follows from Theorem 7 of [5].(Q.E.D.)

The problem of determining when T has continuous spectrum on Γ has now been reduced to the problem of finding, for a prime factor p of n, a condition which is necessary and sufficient for T^p to be strictly ergodic, under the assumption that $p \nmid \lambda_s$ $(s \geqslant 0)$. In order to describe this condition, we require some definitions and new notation.

Let p be a prime factor of n. For $C \in B_k$, $g \in G$, and $0 \leqslant j \leqslant p-1$, we let

$$r_{g,j}(C) = k^{-1}\operatorname{card}\{u : C(u) = g, u \equiv j \pmod p\}.$$

Then clearly,

$$\sum_{j=0}^{p-1} r_{g,j}(C) = r_g(C).$$

If $C \in B_k$, $D \in B_m$, and $k \equiv d \pmod p$, with $d \neq 0$, then the following is easily verified:

$$(1) \quad r_{g,j}(C \times D) = \sum \{r_{e,k}(C) r_{f,m}(D) : e + f = g, k + dm \equiv j \pmod p\}.$$

Now let $G' = G \times Z_p$, and for each $t \geqslant 0$, define an automorphism ϕ_t: $G' \to G'$ as follows:

$$\phi_t(g,j) = (g, \lambda_t^{-1} j)$$

(where λ_t^{-1} refers to the inverse in the field Z_p, which exists by our assumption that $p \nmid \lambda_t$). Let

$$\Phi_{t,k} = \phi_t \circ \phi_{t+1} \circ \cdots \circ \phi_{t+k}.$$

Then

$$\Phi_{t,k}(g,j) = (g, (\lambda_t \lambda_{t+1} \cdots \lambda_{t+k})^{-1} j).$$

For each $t \geqslant 0$, define u_t on G' by

$$u_t(g,j) = npr_{g,j}(b_t).$$

Then from (1), we have

(2) $$u_t * (u_{t+1} \circ \phi_t)(g,j) = npr_{g,j}(b_t \times b_{t+1}),$$

and, in general,

(3) $$u_t * (u_{t+1} \circ \Phi_{t,0}) * \cdots * (u_{t+k} \circ \Phi_{t,k-1})(g,j) = npr_{g,j}(b_t \times \cdots \times b_{t+k}).$$

In order to apply Lemma 2, we define, for any subgroup H of G',

$$H_{t,k} = \Phi_{t,k}(H).$$

Otherwise, notation is as in Lemma 2.

LEMMA 13. *Suppose p is a prime factor of n, and $p \nmid \lambda_s$ $(s \geq 0)$. Then T^p is strictly ergodic if and only if, for every proper subgroup H of $G' = G \times Z_p$,*

$$\lim_{j\to\infty} \left((G':H)^{-j-1} m(u_0,H)m(u_1,H_{0,0}) \cdots m(u_j,H_{0,j-1})\right) = 0.$$

PROOF. Using an argument almost identical to that in the proof of Theorem 4, we see that T^p is strictly ergodic if and only if $r_{g,j}(b_t \times b_{t+1} \times \cdots) = 1/(np)$ for every $g \in G, j \in Z_p$, $t \geq 0$. By (3), this is equivalent to

(4) $$\lim_{j\to\infty} u_t * (u_{t+1} \circ \Phi_{t,0}) * \cdots * (u_{t+k} \circ \Phi_{t,k-1}) = 1 \quad (t \geq 0).$$

If (4) holds for every t, then by looking at the case $t = 0$ and using Lemma 2, we obtain, for any proper subgroup H of G',

$$\lim_{j\to\infty} \left((G':H)^{-j-1} m(u_0,H)m(u_1 \circ \Phi_{0,0},H) \cdots m(u_j \circ \Phi_{0,j-1},H)\right)$$
$$= \lim_{j\to\infty} \left((G':H)^{-j-1} m(u_0,H)m(u_1,H_{0,0}) \cdots m(u_j,H_{0,j-1})\right) = 0.$$

On the other hand, if this limit is 0 for *every* proper subgroup, then for any proper subgroup H, and any $t \geq 0$, let $H' = \Phi_{0,t-1}^{-1}(H)$. We see that for $k \geq 0$, $H'_{0,t+k} = H_{t,k}$. Therefore,

$$\lim_{k\to\infty} \left((G':H)^{-k-1} m(u_t,H)m(u_{t+1},H_{t,0}) \cdots m(u_{t+k},H_{t,k-1})\right) = 0,$$

which implies, by Lemma 2, that

$$\lim_{k\to\infty} u_t * (u_{t+1} \circ \Phi_{t,0}) * \cdots * (u_{t+k} \circ \Phi_{t,k-1}) = 1.$$

(Q.E.D.)

Finally, by combining Theorem 11 and Lemmas 12 and 13, we obtain:

THEOREM 14. *T has continuous spectrum on Γ if and only if, for every prime factor p of n, either $p|\lambda_t$ for infinitely many t, or (if $p \nmid \lambda_s$ for $s > t_0$), for every proper subgroup H of $G' = G \times Z_p$,*

$$\lim_{j\to\infty} \left((G':H)^{-j-1} m(u_{t_0},H)m(u_{t_0+1},H_{t_0,0}) \cdots m(u_{t_0+j},H_{t_0,j-1})\right) = 0.$$

We now consider one special case in which the statement of the theorem is simplified somewhat.

Let $n = 2^k$ for some $k \geqslant 1$; then since 2 is the only prime factor of n, it follows that if every λ_t is odd, all the automorphisms ϕ_t of $G \times Z_2$ are trivial. We may therefore state the result as follows:

COROLLARY 15. *Let $n = 2^k$. T has continuous spectrum on Γ if and only if either λ_t is even for infinitely many t or, for every proper subgroup H of $G' = G \times Z_2$,*

$$\prod_{t=0}^{\infty} \left((G':H)^{-1} m(u_t,H)\right) = 0.$$

In particular, if $n = 2$, there are four proper subgroups of $Z_2 \times Z_2$: $\{(0,0)\}$, $H = \{(0,0),(0,1)\}$, $K = \{(0,0),(1,0)\}$, and $L = \{(0,0),(1,1)\}$. Since H, K, and L all contain the trivial subgroup, we may restrict attention to these three. The statement of Corollary 15 holds automatically for H and K: in the first case because x is strictly transitive, in the second because each b_t is assumed to contain both 0 and 1, and therefore $\frac{1}{2} m(u_t,K) \leqslant 2/3$. It is not hard to see that the statement for the subgroup L is equivalent to condition (2) in Theorem 9 of [5].

4. **Examples.** I. In the case $b_0 = b_1 = \cdots = b$, the sequence x may be obtained from a substitution of constant length (see [10] for the definition). As is observed in [11], for $n = 2$ every sequence arising from a simple substitution can be obtained this way, while for $n > 2$ this is not the case. (The terminology is that of [10]; for $n = 2$, these are usually known as continuous substitutions.)

Michel [7] has shown that any substitution minimal set is strictly ergodic, and Klein [6] obtained this result earlier for substitutions of constant length. For substitution minimal sets which are generalized Morse minimal sets, this is also easily derivable from our Theorem 4. We now consider what can be said in light of Theorem 14 regarding the spectrum of such systems.

For $n = 2$, it is well known [1], and can be derived from [5], that in the case $b_0 = b_1 = \cdots = b$, T has continuous spectrum on Γ. In fact, this can be obtained as a special case of the following result.

THEOREM 16. *If n is prime, and $b_0 = b_1 = \cdots = b$, then T has continuous spectrum on Γ.*

PROOF. If $n | L(b)$, the result is immediate from Theorem 14. We assume $L(b) = \lambda \pmod{n}$, with $\lambda \neq 0$. Then by Fermat's theorem $\lambda^{n-1} = 1 \pmod{n}$; thus by writing $x = B \times B \times \cdots$, where B is the $(n-1)$-fold product $b \times b \times \cdots \times b$, we may assume $\lambda = 1$.

The group G is cyclic, and every proper nonzero subgroup H of $G' = Z_n \times Z_n$ is the cyclic group of order n generated by some element (i,j). If either i or j is 0, the subgroup automatically satisfies the condition in Theorem 14, and it is therefore sufficient to consider subgroups H generated by $(i,1)$ for some $i \neq 0$. Since $\lambda = 1$, all the functions ϕ_t are the identity on G'. Define $u(g,j) = nr_{g,j}(b)$ $((g,j) \in G')$. Then if T does not have continuous spectrum on Γ, Theorem 14 implies that $m(u,H) = 1$ for some H as above; but $r_{0,0}(b) \neq 0$ since $b(0) = 0$, and thus

$$\sum_{(g,j)\in H} r_{g,j}(b) = 1.$$

If $H = \{(0,0),(i_1,1) \cdots (i_{n-1},n-1)\}$, this implies that $b = DD \cdots D0$, where $D = 0i_1i_2 \cdots i_{n-1}$. Now by writing $x = (b \times b) \times (b \times b) \times \cdots$, applying the same argument, and noting that $b \times b$ begins with b, we obtain $b \times b = DD \cdots D0$. Repeating this argument, we see that x must be periodic, which is contrary to our assumption. (Q.E.D.)

For any nonprime n, it is possible to construct examples in which $b_0 = b_1 = \cdots = b$ and T fails to have continuous spectrum on Γ. We give an example for $n = 4$.

Let $G = Z_4$, and let $b = 01032$. Then

$$b \times b = 01032\ 12103\ 01032\ 30321\ 23210,$$

etc. It can easily be seen that $x(i)$ is even if and only if i is even. Thus if we define

$$A = \{y \in \Omega_x : y(0) \text{ is even}\}, \qquad A' = \Omega_x \setminus A,$$

the function $f = 1_A - 1_{A'}$ satisfies $Tf = -f$; since $-1 \not\in \Delta$, $f \in \Gamma$. (It can also be verified that the condition in Corollary 15 fails for the subgroup H of $Z_4 \times Z_2$ generated by (1,1).)

II. For our final example, again we take $G = Z_4$. Let $b_0 = 33031$, $b_1 = 01332$, $b_2 = 02122$, $b_3 = 03312$, $b_4 = 00102$, and $b_k = b_{k+4}$ $(k \geqslant 1)$. Then it is clear that (Ω_x, T) is strictly ergodic by Theorem 4. To verify that T has continuous spectrum on Γ, it is sufficient to consider the subgroups H_1 and H_2 of $Z_4 \times Z_2$, where H_1 is generated by (1,1) and $H_2 = \{0,2\} \times Z_2$. It can be verified that the condition in Corollary 15 is satisfied.

If we identify the sequence x with a sequence y on the four symbols 2,4,6,8 by the rule $0 \to 2$, $1 \to 4$, $2 \to 8$, $3 \to 6$, then it can be shown that y is the sequence constructed by Kakutani [4, Example 2]. There it is constructed from the formula

$$y(k) = \text{the last nonzero digit in the decimal expansion of } k! \ (k \geqslant 2).$$

Thus our Theorems 4 and 14 generalize Theorem 3 and a portion of Theorem 4 of [4].

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